

Volatility Exposure for Strategic Asset Allocation

M. Brière, A. Burgues and O. Signori

This paper examines the advantages of incorporating equity volatility exposure into a global balanced portfolio. We consider two sets of strategies: long implied volatility and long volatility risk premium strategies. To calibrate and assess the risk/return profile of the portfolio, we present an analytical framework, offering pragmatic solutions for long-term investors seeking exposure to volatility. The benefit of volatility exposure for a conventional portfolio is shown through a mean / Value-at-Risk portfolio optimization. Pure volatility investment makes it possible to partially hedge downside equity risk, thus reducing the risk profile of the portfolio. Investing in the volatility risk premium substantially increases returns for a given level of risk. A well calibrated combination of the two strategies allows for enhanced absolute and risk-adjusted returns for the portfolio.

Keywords : variance swap, volatility risk premium, higher moments, portfolio choice, Value at Risk.

JEL codes : G11, G12, G13

CEB Working Paper N° 08/034
September 2008

Volatility Exposure for Strategic Asset Allocation

M. Brière^{1,2}, A. Burgues² and O. Signori^{2,*}

14 November 2008

Abstract

This paper examines the advantages of incorporating equity volatility exposure into a global balanced portfolio. We consider two sets of strategies: long implied volatility and long volatility risk premium strategies. To calibrate and assess the risk/return profile of the portfolio, we present an analytical framework, offering pragmatic solutions for long-term investors seeking exposure to volatility. The benefit of volatility exposure for a conventional portfolio is shown through a mean / Value-at-Risk portfolio optimization. Pure volatility investment makes it possible to partially hedge downside equity risk, thus reducing the risk profile of the portfolio. Investing in the volatility risk premium substantially increases returns for a given level of risk. A well calibrated combination of the two strategies allows for enhanced absolute and risk-adjusted returns for the portfolio.

Keywords : variance swap, volatility risk premium, higher moments, portfolio choice, Value at Risk.
JEL codes : G11, G12, G13

¹ Centre Emile Bernheim
Solvay Business School
Université Libre de Bruxelles
Av. F.D. Roosevelt, 50, CP 145/1
1050 Brussels, Belgium.

² Crédit Agricole Asset Management,
90 bd Pasteur, 75015 Paris

*Comments could be sent to marie.briere@caam.com, alexandre.burgues@caam.com, or ombretta.signori@caam.com. The authors are grateful to E. Bourdeix, T. Bulger, F. Castaldi, B. Drut, C. Guillaume, B. Houzelle, S. de Laguiche, A. Szafarz, K. Topeglo for helpful comments and suggestions.

Direct exposure to volatility has been made easier, for a wide range of underlyings, by the creation of standardized instruments. The widespread use and increasing liquidity of volatility index futures and variance swaps clearly show that investors are taking a keen interest in volatility. In addition to short-term trading ideas, some investors now look for a structural exposure to volatility as they consider it either as a well identified asset class or, at the very least, a set of strategies with strong diversifying potential for their portfolios.

The scope of this paper is therefore to construct an analytical framework useful for investors willing to assess the benefits of a long-term investment in volatility. Exposure to volatility is achieved, in our case, through the combination of two very different sets of strategies: on the one hand, long investment in implied volatility and, on the other, long exposure to the volatility risk premium, the latter defined as the difference between the implied volatility of an underlying and its subsequent realized volatility. Both sets of strategies are consistent with the classic motivations that push investors to move into an asset class, i.e. the possibility for diversification and return enhancement. The remarkably strong negative correlation between implied volatility and equity prices during market downturns offers timely protection against the risk of capital loss. So being long implied volatility is highly attractive to investors for diversification purposes. Additionally, exposure to the volatility risk premium has historically delivered attractive risk-adjusted returns albeit with greater downside risk. Investing in the volatility premium is a strategy similar to selling insurance premiums.

Although the two approaches differ in terms of rational, they both raise the problem of how to measure a portfolio's expected utility when returns are non-normal. The issue is relevant as returns of volatility strategies are asymmetric and leptokurtic. Accordingly, the goal of minimizing risk through a conventional

mean-variance optimization framework can be misleading as extreme risks are not properly captured. (Sornette *et al.* (2000)). For volatility premium strategies, low volatility of returns is generally countered by higher negative skewness and kurtosis, which could turn costly for investors if not properly taken in account (Amin and Kat (2003)). Thus, it is necessary to use optimization techniques in order to assess risk through measures which capture higher-order moments of the return distribution: Value-at-Risk is a better tool for our purposes (Favre and Galeano (2002), Agarwal and Naik (2004), Martellini and Ziemann (2007)). Another aspect that needs to be addressed is the practical implementation of exposing a portfolio to volatility. Because these strategies are implemented through derivative products, they require limited capital, leverage being the key factor. In this case, the amount of risk to be taken in the portfolio needs to be properly calibrated.

To our knowledge, adding volatility exposure to a balanced portfolio, as described above, is an exercise yet to be assessed. Daigler and Rossi (2006) analyzed the effect of adding a long volatility strategies to an equity portfolio, while Dash and Moran (2005) looked at the impact on funds of hedge funds. Hafner and Wallmeier (2007) examined the contribution to an equity portfolio of a volatility risk premium strategy. But all these papers have in common the use of the mean-variance framework when optimizing the portfolio composition.

This paper takes the case of an investor managing a classic balanced portfolio composed of equities and government bonds, and looking to add a strategic exposure to equity volatility. To take into account the particular shape of volatility strategies' return distributions, efficient frontiers are built within a Mean / Value-at-Risk framework. We look at how to optimize the combination of these strategies and how they impact the portfolio's long term performance.

We believe this research is original for two reasons. First, it offers a framework for

analyzing the inclusion of volatility strategies into a portfolio; second, it combines two very different sets of volatility exposures: a) long implied volatility and b) long volatility premium. We show how this combination can deliver compelling results for the portfolio, not just in terms of enhanced returns but also in terms of risk diversification. Adding long implied volatility leads to lower levels of risk since this strategy typically hedges downside equity risk. Meanwhile, investing in the volatility risk premium enhances returns for a given level of risk. Finally, a combination of the two strategies is also appealing because they tend to hedge each other in adverse events.

INVESTING IN VOLATILITY

We begin by presenting two complementary approaches to investing in volatility. We then look at using these strategies to build a portfolio.

Long exposure to volatility

One approach is to expose a portfolio to implied volatility changes in an underlying asset. The rationale for this kind of investment is primarily the diversification benefits arising from the strongly negative correlation between performance and implied volatility of the underlying, particularly during a bear market (Daigler and Rossi (2006)).

Tracking the implied volatility for an underlying requires a synthetic volatility indicator. A volatility index, expressed in annualized terms, prices a portfolio of options across a wide range of strikes (volatility skew) and with constant maturity (interpolation on the volatility term structure). Within the family of volatility indices, the Volatility Index (VIX) of the Chicago Board Options Exchange (CBOE) is widely used as a benchmark by investors. The VIX is the expression of the 30-day implied volatility generated from S&P 500 traded options. The details of the index calculation are given in a *White Paper* published by the CBOE

(2004)¹. Because it reflects a consensus view of short-term volatility in the equity market, the VIX (see the time series plotted in Figure 1 of Appendix 1) is widely used as a measure of market participants' risk aversion (the "investor fear gauge").

Although the VIX index itself is not a tradable product, the Chicago Futures Exchange² launched futures contracts on the VIX in March 2004. Thus investors now have a direct way of exposing their portfolios to variations in the short-term implied volatility of the S&P 500. VIX futures provide a better alternative to achieving such exposure than traditional approaches relying on the use of delta-neutral combinations of options such as straddles (at-the-money call and put), strangles (out-of-the-money call and put) or more complex strategies (such as volatility-weighted combinations of calls and puts). On short maturities (less than 3 months), the impact of neutralizing the delta exposure of these portfolios can easily dominate the impact of implied volatility variations.

The approach we take in establishing a structurally long investment in implied volatility tries to take advantage of the mean-reverting nature of volatility³ (Dash and Moran (2005)). This is achieved by calibrating the exposure according to the absolute levels of the VIX, with the highest exposure when implied volatility is at historical low levels, and reducing such exposure as volatility rises. Implementing the long volatility (LV) strategy then consists in buying the correct number of VIX futures such that the impact of a 1-point variation in the price of the future is equal to $\frac{1}{F_{t-1}} * 100\%$ (5% impact when the level of VIX is 20). The P&L

generated between $t-1$ (contract date) and t (maturity date) can then be written as:

$$PL_t^{VIX} = \frac{1}{F_{t-1}} (F_t - F_{t-1}) \quad (1)$$

Where F_t is the price of the future at time t .

In practice, VIX futures prices are available only since 2004. They represent the 1-month forward market price for 30-day implied volatility. This forward-looking component is reflected in the existence of a term premium between the VIX future and VIX index. This premium tends to be positive when volatility is low (it represents a cost of carry for the buyer of the future) and negative when volatility peaks. We approximated VIX futures prices prior to 2004 using the average relationship between VIX futures and the VIX index, estimated econometrically over the period from March 2004 to August 2008⁴.

Capturing the volatility risk premium

The second volatility strategy involves taking exposure to the difference between implied and realized volatility. This difference is defined as a risk premium and has historically been positive on average for equity indices (Carr and Wu (2007)). This volatility risk premium (VRP), well documented in the literature (Bakshi and Kapadia (2003), Bondarenko (2006)), can be explained by the risk asymmetry between a short volatility position (net seller of options faces an unlimited potential loss), and a long volatility position (where the loss is capped at the amount of premium paid). To make up for the uncertainty on the future level of realized volatility, sellers of implied volatility demand compensation in the form of a premium over the expected realized volatility⁵.

¹ The method of calculation has evolved in September 2003. The current method (applied retroactively to the index since 1990) takes into account the S&P500 traded options at all strikes, unlike the previous VXO index was based solely on at-the-money S&P 100 options.

² Part of the Chicago Board of Options Exchange.

³ Empirical tests have shown that having an exposure inversely proportional to the observed level of implied volatility markedly increases the profitability of the strategy.

⁴ Multiplied by a factor of 10 in accordance with the CBOE methodology.

⁵ Other components can provide partial explanations of this premium: the convexity of the P&L of the variance swap, and the fact that investors tend to be structural net buyers of volatility (Bollen and Whaley (2004)).

The VRP (see Figure 2 in Appendix 1 for a historical time series) is captured by investing in a variance swap, i.e. a swap contract on the spread between implied and realized variance. Through an over-the-counter transaction, the two parties agree to exchange a specified implied variance level for the actual amount of variance realized over a specified period. The implied variance at inception is the level that gives the swap a zero net present value. From a theoretical point of view, this level (or strike) is computed from the price of the portfolio of options that is used to calculate the volatility index itself. Thus, the theoretical strike for a 1-month variance swap on the S&P 500 is the value of the VIX index. In practice, however, owing to the difficulty of replicating the index, it is more realistic to reduce VIX implied volatility by 1% to reflect the costs of replication borne by arbitrageurs (Standard & Poors (2008)).

We consider a short variance swap strategy on the S&P 500 held over a one month period. The P&L of a short variance swap position between the start date ($t-1$) and end date (t) can be written as:

$$PL_t^{VARSWAP} = \frac{N_{VEGA}}{2K_{t-1}} * [K_{t-1}^2 - RV_{t-1,t}^2] \quad (2)$$

Where K_{t-1} is the volatility strike of the variance swap contract entered at date $t-1$, $K_t = \frac{VIX_t}{100}$, VIX_t is the VIX index, $RV_{t-1,t}$ is the realized volatility between $t-1$ and t , and N_{VEGA} is the vega notional of the contract (see Appendix 2 for further details on variance swaps).

Adequate risk measure

An important aspect of implementing volatility strategies is the non-normality of return distributions, as shown in the next section. When returns are not normally distributed, the mean-variance criterion of Markowitz (1952) is no longer adequate. To compensate for this, many authors have sought to include higher-

order moments of the return distribution in their analysis. Lai (1991) and Chunchachinda et al. (1997), for example, introduce the third moment of the return distribution (ie skewness) and show that this leads to significant changes in the optimal portfolio construction. Extending portfolio selection to a four-moment criterion brings a further significant improvement (Jondeau and Rockinger (2006, 2007)).

Within the proposed volatility investment framework, the main danger for the investor is the risk of substantial losses in extreme market scenarios (left tail of the return distribution). As returns on volatility strategies are not normally distributed, we choose “modified Value-at-Risk” as our reference measure of risk. Value-at-Risk (VaR) is defined as the maximum potential loss over a time of period given a specified probability α . Within normally distributed returns, VaR can be written:

$$VaR(1-\alpha) = \mu - z_\alpha * \sigma \quad (3)$$

Where μ and σ are, respectively, the mean and standard deviation of the return distribution and z_α is the α -quantile of the standard normal distribution $N(0,1)$.

To capture the effect of non-normal returns, we replace the quantile of the standard normal distribution with the “modified” quantile of the distribution w_α , approximated by the Cornish-Fisher expansion based on a Taylor series approximation of the moments (Stuart, Ord and Arnold (1999)). This enables us to correct the distribution $N(0,1)$ by taking skewness and kurtosis into account. Modified VaR is accordingly written as:

$$ModVaR(1-\alpha) = \mu - w_\alpha * \sigma \quad (4)$$

Where

$$w_\alpha = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1) * S + \frac{1}{24}(z_\alpha^3 - 3z_\alpha) * EK - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha) * S$$

w_α is the modified percentile of the distribution at threshold α , S is the skewness and EK is the excess kurtosis of the portfolio.

Two portfolios that offer the same expected return for a given level of volatility will have the same “normal” VaR but different modified VaR if their returns present different skewness or/and excess kurtosis. In particular, modified VaR will be greater for the portfolio that has negative skewness (left-handed return distribution) and/or higher excess kurtosis (leptokurtic return distribution). In addition to being simple to implement in the context of constructing the investor’s risk budget, modified VaR explicitly takes into account how the investor’s utility function changes in the presence of returns that are not normally distributed. A risk-averse investor will prefer a return distribution where the odd moments (expected return, skewness) are positive and the even moments (variance, kurtosis) are low.

Calibrating volatility strategies

In practice, when volatility strategies are implemented, cash requirements are very limited as the required exposure is achieved through listed or OTC derivatives. The only capital requirement is, in fact, the collateral needed when entering a Variance Swap contract, and futures margin deposits for listed products. A key step in the process of investing in the volatility asset class is the proper calibration of the strategies.

Each volatility strategy is calibrated according to the maximum risk exposure allowed. Based on our computations of modified VaR for each asset class, we set monthly modified 95% VaR at 6%, ie at levels comparable to the equity asset class (see Table 1, Appendix 3). The volatility strategies’ returns are thus the return on cash plus a fixed proportion (that we call for simplicity “degree of leverage”) of the strategy’s P&L, this proportion being determined ex ante by our calibration on the back of the allowed risk:

$$r_t^{LV} = r_t^f + L_1 * PL_t^{VIX} \quad (5)$$

$$r_t^{VRP} = r_t^f + L_2 * PL_t^{VARSWAP} \quad (6)$$

Where r_t^{LV} (resp. r_t^{VRP}) is the monthly return of the LV strategy (resp. VRP), r_t^f is the cash return, L_1 (resp. L_2) is the degree of leverage calibrated on the LV strategy (resp. VRP).

BUILDING AN EFFICIENT PORTFOLIO WITH VOLATILITY

We consider the simple case of a long-term portfolio invested in equities and US government bonds, to which we add our two volatility strategies.

Data

For equities, we use the S&P 500 index, and for government bonds the 7-10 year Merrill Lynch index. For the volatility strategies, we use the VIX index of the CBOE. For the risk-free rate, we consider the 1-month US interbank rate⁶. The study period runs from February 1990 to August 2008.

Summary statistics

Table 1 in Appendix 3 presents the statistics of the four “assets” included in our study. Looking at Sharpe ratios and success rates⁷, the VRP strategy appears to be the most attractive, with a Sharpe ratio of 2.8 and a success rate of 85%. Bonds (0.5 and 68%), equities (0.4 and 64%) and LV strategy (0.2 and 53%) follow in that order. As we will show, although the LV strategy is last in this ranking, it holds considerable interest in terms of diversification power. The VRP strategy, on the other hand, is the most consistent winner. It has a relatively stable performance, the exception being during periods of rapidly increasing realized volatility (onset of crises, unexpected market shocks), when returns are strongly negative⁸ and much greater in amplitude than for the traditional asset classes. These periods are

⁶ All of the data were downloaded as monthly series from Datastream.

⁷ See Grinold & Kahn (2000) for the relation between the Sharpe ratio and the success rate.

⁸ Realized volatility rises above implied volatility.

usually of short duration, accounting for only 15% of the months in the studied period.

Given the calibration chosen, the volatility strategies have the highest volatility (18% for LV, 17% for VRP) – higher than the one on equities (14%) and almost three times that on government bonds (6%). Downside deviation, a measure of the asymmetric risk on the left side of the return distribution, offers a slightly different picture: equities and the VRP strategy appear riskier than the LV strategy. Monthly mean returns range between 0.58% for LV and 3.59% for the VRP strategy. Analysis of extreme returns (min and max) highlights the asymmetry of the two volatility strategies: the LV strategy offers the highest maximum return at 26.52% (its minimum return at -10.41%), whereas the VRP strategy posts the worst monthly performance at -26.31% (with the best month at 15.28%).

The higher-order moments clearly highlight that returns are not normally distributed⁹, especially for the two volatility strategies. The skewness of equity and bond returns is slightly negative (-0.46 and -0.31 respectively), and for the VRP strategy it shows a very strong negative figure (-1.81). The only strategy showing positive skewness (1.00) is LV: being long implied volatility provides a partial hedge of the leftward asymmetry of the other asset classes. All four assets have kurtosis greater than 3: 3.54 and 3.86 for bonds and equities and even higher for the volatility strategies: 5.34 (LV) and 10.37 (VRP).

Codependencies

The multivariate characteristics of returns are likewise of great interest. The correlation matrices are shown in Table 2 of Appendix 3. For the 1990–2008 period, we find good diversification power between equities and bonds, in the form of virtually zero correlation. As expected, the LV strategy offers strong

⁹ For equity returns and returns on the two volatility strategies, the null hypothesis of a normality test is significantly rejected.

diversification power relative to the traditional asset classes. It is highly negatively correlated with equities (-61%), a phenomenon already well publicized by other studies (Daigler and Rossi (2006)). Less well known, the LV strategy is also weakly correlated with bonds, at 8%. The VRP strategy shows quite different characteristics: it offers little diversification to equity exposure (46% correlation), but significantly more to bonds (-17%). More importantly, the two volatility strategies are mutually diversifying (-61% correlation), and this, as we will see, is of great interest for portfolio construction.

The importance of extreme risks also requires the analysis of the coskewness and cokurtosis matrices of the asset classes (Tables 3 and 4¹⁰ in Appendix 3). Positive coskewness value sk_{ij} ¹¹ suggests that asset j has a high return when volatility of asset i is high, i.e., j is a good hedge against an increase in the volatility of i . This is particularly true for the LV strategy, which offers a good hedge for the VRP strategy, and in a less extent for equities and bonds. In contrast, the VRP strategy does not hedge the other assets efficiently because it tends to perform poorly when their volatility increases.

Positive cokurtosis value ku_{ij} ¹² means that the return distribution of asset i is more negatively “skewed” when the return on asset j is lower than expected, ie i is a poor hedge against a decrease in the value of j . Here again, we find that the LV strategy is an excellent hedge against equities (far better than a long bond), unlike the VRP strategy. However, the two volatility strategies hedge each other quite well. Positive cokurtosis ku_{ijk} is a sign that the

¹⁰ We give a summary presentation of these matrices. For $n=4$ assets, it suffices to calculate 20 elements for the coskewness matrix of dimension (4,16) and 35 elements for the cokurtosis matrix of dimension (4, 64).

¹¹ The general formula for coskewness is:

$$sk_{ijk} = \frac{E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)]}{\sigma_i \sigma_j \sigma_k},$$

where r_i is the return on asset i and μ_i its mean.

¹² The general formula for cokurtosis is:

$$ku_{ijkl} = \frac{E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)(r_l - \mu_l)]}{\sigma_i \sigma_j \sigma_k \sigma_l}$$

covariance between j and k increases when the volatility of asset i increases. The most interesting results are delivered in periods of rising equity volatility. The LV/bonds correlation increases, whereas the VRP/bonds and VRP/LV correlations decline. Thus, during periods of stress in the equity market, VRP and equities perform badly at the same time, while LV and bonds do better. Lastly, positive cokurtosis ku_{ijj} means that volatilities of i and j tend to increase at the same time. This is the case for all four assets. Here again, all coskewness and cokurtosis values are respectively significantly different from 0 and 3, a sign that the structure of dependencies between these strategies differs significantly from a multivariate normal distribution.¹³

This first analysis already allows us to show different advantages of the two volatility strategies within a diversified portfolio: the LV strategy delivers excellent diversification relative to equities and, to a lesser extent, bonds; the VRP strategy allows for very substantial increase in returns, at the expense of a broadly increased risk profile (extreme risks and codependencies with equities). A combination of the two volatility strategies appears particularly attractive since they tend to hedge each others' risks, especially in extreme market scenarios.

Efficient portfolios

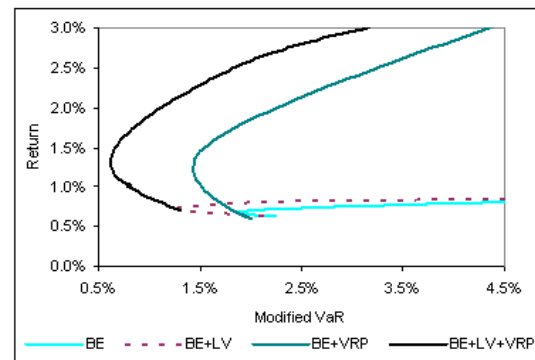
As previously noted, in our analytical framework, the two volatility strategies are collateralized (a fixed amount of cash, 6%, is used for collateral and margin purposes). To construct the portfolio, the sum of the percentage shares in the four asset classes must equal 100%. For the two traditional asset classes (equities and bonds), short sales are not allowed (long-only portfolio). For the two volatility strategies implemented via derivatives, long and short positions are allowed.

¹³ The null hypothesis of a multivariate normality test (Kotz et al. (2000)) is significantly rejected.

We compute efficient frontiers in a mean-VaR framework by considering: (1) the initial portfolio invested 100% in equities and government bonds, the initial portfolio with the addition of (2) the LV strategy, (3) the VRP strategy and (4) the two volatility strategies at the same time.

Figure 3 shows the four efficient frontiers. A first observation is that adding the volatility strategies markedly improves the efficient frontier compared with the initial portfolio of equities and bonds.

Figure 3: Efficient Frontiers for the Four Portfolios, Entire Period (February 1990 – August 2008)



For a detailed examination of our results, we first look at portfolio performances minimizing VaR exposure. The corresponding allocations are presented in Table 5 (Appendix 4). Compared with the initial portfolio (79% bonds, 21% equities), the addition of the LV strategy (19%) combined with an increase in the allocation to equities (31%) and a decrease in bonds (50%) reduces the VaR to 1.3% from 1.9%. The resulting portfolio has greater interest for the investor, in the form of a higher Sharpe ratio (0.9 versus 0.7) obtained through an increase in annualized return (8.7% versus 8.2%) and a decrease in volatility (4.7% versus 5.4%). This result is mostly due to the strong negative correlation between the LV strategy and equities (-61%). Furthermore, the distribution of returns for the new portfolio shows a notable improvement of the higher-order moments. The portfolio offers positive skewness (+0.47 versus virtually nil for the

initial portfolio) and an overall decrease in kurtosis from 4.15 to 3.93.

Adding the VRP strategy (20%) to the initial portfolio, at the expense of equities (2%) and, to a lesser extent, bonds (78%), makes it possible to achieve significantly higher returns (14.89% versus 8.49%), along with a lower VaR (1.4%). The success rate of the portfolio is improved to 80.3%, and the Sharpe ratio rises to 2. The portfolio return distribution shows a more pronounced leftward asymmetry (-0.41 versus -0.04), which reduces its appeal for the most risk-averse investors.

Finally, the most interesting risk/return profile is obtained by adding a combination of the two volatility strategies. Adding both the LV (24%) and the VRP strategy (20%), at the expense of bonds (40%) and equities (16%) makes it possible to achieve a VaR of 0.6%. The success rate increases significantly, and the Sharpe ratio (2.72) is the highest of all of the four portfolios. The decrease in extreme risks is reflected in the higher-order moments. Compared with the initial one, this combined portfolio presents the same asymmetry to the left (skewness of -0.04) and is slightly more leptokurtic (kurtosis of 4.24 versus 4.15). Downside risk as measured by the worst month performance improves to -3.19% from -4.02% .

Another way of looking at the results is to compare optimal portfolios with identical VaR. We examined two cases that provide a fairly representative picture of the set of achievable portfolios. In the case of 2% VaR (Table 6, Appendix 4), the classic optimized bonds/equities portfolio is invested 69% in bonds and 31% in equities. It achieves a return of 8.5% and a Sharpe ratio of 0.7. Adding volatility strategies improves performance significantly. Adding only the LV strategy, it is possible to achieve a return of 9.54% (Sharpe ratio of 0.79, obtained with 56% equities, 6% bonds and 38% volatility). Combining the traditional portfolio with the VRP strategy produces an even better return (22.3%, Sharpe ratio of 2.49); in this case, the optimal portfolio no longer contains directional equity exposure

(highly correlated with variance swaps), but instead consists of 60% bonds and 40% VRP. Finally, the most attractive portfolio is obtained by combining the two volatility strategies. With 38% LV strategy and 62% VRP, it is possible to achieve a return of 30.89% (Sharpe ratio of 3.10). The two traditional asset classes disappear in this optimal portfolio.

If we consider the case of a portfolio with VaR of 4% (Table 7 in Appendix 4), all the portfolios contain significantly more of the riskier assets. The first two portfolios (bonds/equities and bonds/equities/LV) contain a far higher proportion of equities (70% and 84% respectively). The last two contain a markedly higher proportion of the VRP strategy (72% and 84% respectively). In every case, the portfolio's return is higher but the Sharpe ratio is lower, since the increase in returns does not completely offset the increase in risk. It is significant that, in these last two examples (VaR of 2% and 4%), the optimal portfolios no longer contain any equities or bonds, but only the two volatility strategies. These are "extreme" portfolios that take on levels of risk (and leverage) close to those established in the hedge fund industry.

CONCLUSION

Recent literature has begun to show the merit of including long exposure to implied volatility in a pure equity portfolio (Daigler and Ross (2006)) or in a portfolio of funds of hedge funds (Dash and Moran (2006)). The purpose of this paper was to examine portfolios that combine classic asset classes (equity and bonds) with a combination of volatility strategies, as little has been written so far on the subject. Among the simplest strategies for adding volatility exposure to a traditional portfolio, we identified not only buying implied volatility but also investing in the volatility risk premium. While these strategies have attracted considerable interest on the part of some market professionals, especially hedge fund managers (and, more recently, more sophisticated

managers of traditional funds), the academic literature to date has paid little attention to them.

We explored a long exposure to two very simple types of volatility strategy added to a diversified bonds/equities portfolio. Our results from a historical analysis of the past twenty years show the great interest of including these volatility strategies in such a portfolio. Taken separately, each of the strategies displaces the efficient frontier significantly outward, but combining the two produces even better results. A long exposure to volatility is particularly valuable for diversifying a portfolio holding equities: because of its negative correlation to the asset class, its hedging function during bear-market periods is clearly interesting. For its part, a volatility risk premium strategy boosts returns. It provides little diversification to the equities (it loses significantly when equities fall) but rather good diversification with respect to bonds and implied volatility. Combining the two strategies offers the big advantage of fairly effective reciprocal hedging during periods of market stress, significantly improving portfolio return for a given level of risk.

One of the limits of our work relates to the period analyzed. Although markets experienced several important crises over the period from 1990 to 2008, with important volatility spikes, there is no assurance that, in the future, crises will not be more acute than those experienced over the testing period and that losses on variance swaps positions will not be greater, thereby partly erasing the high reward associated with the volatility risk premium. An interesting continuation of this work would be to explore the extent to which long exposure to volatility is a satisfactory hedge of the volatility risk premium strategy during periods of stress and sharp increases in realized volatility.

Like fixed-income and equity markets, volatility as an asset class can be approached not only in terms of directional volatility strategies but also in terms of inter-class arbitrage strategies (relative value, correlation trades, etc.). Tactical strategies can also be envisaged.

The possibilities are numerous, and they deserve further investigation to precisely measure both the benefits and the risks to an investor who incorporates such strategies in an existing portfolio.

BIBLIOGRAPHY

Agarwal V. and Naik N. (2004), "Risks and Portfolio Decisions Involving Hedge Funds." *Review of Financial Studies*, 17(1), p. 63–8.

Allen P., Einchcomb S. and Granger N. (2006), "Variance Swaps." *JP Morgan European Equity Derivatives Strategy*, 14 November.

Bakshi G. and Kapadia N. (2003), "Delta-Hedged Gains and the Negative Market Volatility Risk Premium." *The Review of Financial Studies*, 16 (2), p. 527–566.

Bollen N.P.B. and Whaley R.E. (2004), "Does Net Buying Pressure Affect the Shape of Implied Volatility Functions." *The Journal of Finance*, 59(2), p. 711–753.

Bondarenko O. (2006), "Market Price of Variance Risk and Performance and Hedge Funds." *AFA 2006 Boston Meetings Paper*.

Carr P. and Wu L. (2007), "Variance Risk Premia." *AFA Philadelphia Meetings*, October.

CBOE (2004), "VIX CBOE Volatility Index." *Chicago Board Options Exchange Web site*.

Chunhachinda P., Dandapani K., Hamid S. and Prakash A.J. (1997), "Portfolio Selection With Skewness: Evidence from International Stock Markets." *Journal of Banking and Finance*, 21(2), p. 143–167.

Daigler R.T. and Rossi L. (2006), "A Portfolio of Stocks and Volatility." *The Journal of Investing*, Summer.

Dash S. and Moran M.T. (2005), *VIX as a Companion for Hedge Fund Portfolios*, *The Journal of Alternative Investments*, Winter.

Demeterfi K., Derman E., Kamal M. and Zhou J. (1999), "A Guide to Volatility and Variance Swaps." *The Journal of Derivatives*, Summer.

Favre L. and Galeano J.A. (2002), "Mean-modified Value at Risk Optimization With

Hedge Funds." *Journal of Alternative Investment*, Fall.

Guidolin M. and Timmerman A. (2005), "Optimal Portfolio Choice under Regime Switching, Skew and Kurtosis Preferences." *Working Paper*, Federal Reserve Bank of St. Louis.

Grinold & Khan (2000) , "The Fundamental Law of Active Management », McGraw-Hill.

Hafner R., Wallmeier M. (2007), "Volatility as an Asset Class: European Evidence." *The European Journal of Finance*, 13(7), October.

Harvey C., J. Liechty, M. Liechty and P. Müller (2003), "Portfolio Selection with Higher Moments." *Working Paper*, Duke University.

Jondeau E. and M. Rockinger (2006), "Optimal Portfolio Allocation under Higher Moments." *Journal of the European Financial Management Association* 12, p. 29–55.

Jondeau E. and M. Rockinger (2007), "The Economic Value of Distributional Timing." *Swiss Finance Institute Research Paper* 35.

Kotz S., Balakrishnan N. and Johnson N.L. (2000), "Continuous Multivariate Distributions, Volume 1: Models and Applications, John Wiley, New York.

Lai T.Y. (1991), "Portfolio Selection with Skewness: A Multiple Objective Approach." *Review of Quantitative Finance and Accounting*, 1, p. 293–305.

Markowitz H., (1952), "Portfolio Selection." *Journal of Finance* 7(1), p. 77–91.

Martellini L. and Ziemann V. (2007), "Extending Black-Litterman Analysis Beyond the Mean-Variance Framework." *Journal of Portfolio Management*, Summer.

Sornette D., Andersen J.V. and Simonetti P. (2000), "Portfolio Theory for Fat Tails."

International Journal of Theoretical and Applied Finance, 3(3), p. 523–535.

Standard & Poors (2008), “S&P500 Volatility Arbitrage Index: Index Methodology.” January.

Stuart A., Ord K. and Arnold S.(1999), “Kendall’s Advanced Theory of Statistics, Volume 1 : Distribution Theory.” 6th edition, Oxford University Press.

Appendices

Appendix 1: Implied volatility and volatility risk premium

Figure 1: Implied Volatility (VIX), February 1990 – August 2008

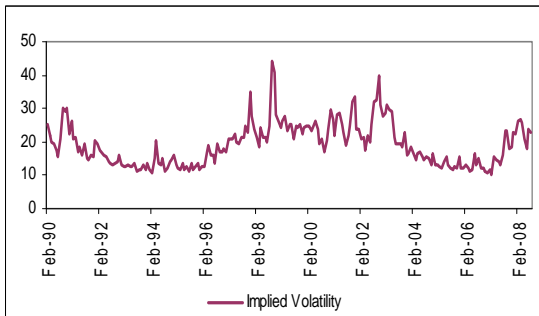
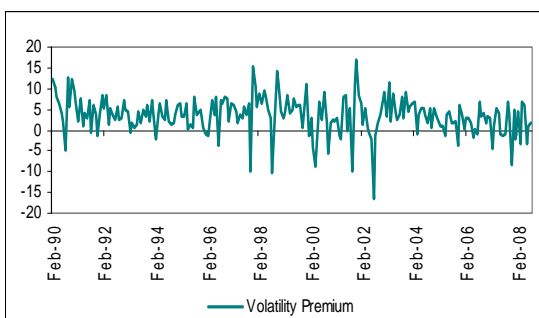


Figure 2: Volatility Risk Premium, February 1990 – August 2008



Appendix 2: How a Variance Swap Works

From a theoretical standpoint, a variance swap can be seen as a representation of the structure of implied volatility (the volatility “smile”) since the strike price of the swap is determined by the prices of options of the same maturity and different strikes (all available calls/puts in, at, or out of the money) that make up a static portfolio replicating the payoff at maturity. The calculation methodology for the VIX volatility index represents the theoretical strike of a variance swap on the S&P 500 index with a maturity of one month (interpolated from the closest maturities so as to keep maturity constant).

From a practical standpoint, the two markets are closely linked through the hedging activity of

market-makers: to a first approximation, a market-maker that sells a variance swap will typically hedge the *vega* risk on its residual position by buying the 95% out-of-the-money put on the listed options market.

The P&L of a variance swap is expressed as follows (Demeterfi et al. (1999)):

$$P \& L = N_{\text{variance}} * [RV_{0,T}^2 - K_T^2]$$

Where K_T is the volatility strike of a variance swap of maturity T (K_T^2 is the delivery price of the variance), $RV_{0,T}$ is the realized volatility of the asset underlying the variance swap over the term of the swap, and N_{variance} is the “variance notional.”

Realized volatility $RV_{0,T}$ is calculated from closing prices of the S&P 500 index according to the following formula:

$$RV_{0,T} = \sqrt{\frac{252}{T} \sum_{t=1}^T \left[\ln \left(\frac{SP500_t}{SP500_{t-1}} \right) \right]^2}$$

In terms of the Greek-letter parameters popularized by the Black-Scholes-Merton option pricing model, the notional of a variance swap is expressed as a vega notional, which represents the mean P&L of a variation of 1% (one vega) in volatility. Although the variance swap is linear in variance, it is convex in volatility (a variation in volatility has an asymmetric impact). The relationship between the two notionals is the following:

$$N_{\text{vega}} = N_{\text{variance}} * 2K$$

Where N_{vega} is the vega notional.

Appendix 3: Descriptive Statistics

Table 1
Descriptive Statistics
Monthly Returns, US, February 1990 – August 2008

	Geometric Mean	Ann. Geometric Mean	Median	Min	Max	Ann. Std. dev.	Skewness	Kurtosis	Ann. Down. dev.*	Mod. VaR	Sharpe Ratio	Success Rate
Bond	0.62%	7.68%	0.61%	-5.55%	5.38%	5.84%	-0.31	3.54	2.97%	2.27%	0.53	68%
Equity	0.79%	9.89%	1.28%	-14.46%	11.44%	13.71%	-0.46	3.86	7.81%	6.07%	0.39	64%
LV	0.58%	7.24%	0.18%	-10.41%	26.52%	18.18%	1.00	5.34	8.07%	6.00%	0.15	53%
VRP	3.59%	52.75%	4.20%	-26.31%	15.28%	17.27%	-1.81	10.37	9.74%	6.00%	2.79	85%

*Downside Deviation is determined as the sum of squared distances between the returns and the cash return series.

Table 2
Correlation matrix
Monthly Returns, US, February 1990–August 2008

	Bonds	Equity	LV	VRP
Bonds				
Equity	-0.01			
LV	0.08	-0.61		
VRP	-0.17	0.46	-0.61	

Table 3
Co-Skewness matrix
Monthly Returns, US, February 1990 – August 2008

	Bonds^2	Equity ^2	LV^2	VRP^2	Bonds*Equity	Bonds*LV	Equity*LV
Bonds	-0.31	0.35	0.04	0.46			
Equity	-0.03	-0.46	-0.67	-0.85			
LV	0.20	0.59	1.00	0.89	-0.13		
VRP	-0.14	-0.50	-0.72	-1.81	0.22	-0.21	0.57

Table 4
Co-kurtosis matrix
Monthly Returns, US, February 1990 – August 2008

	Bonds^3	Equity ^3	LV^3	VRP^3	Bonds* Equity^2	Bonds* LV^2	Bonds* VRP^2
Bonds	3.54	-0.53	0.98	-2.49	1.53	1.39	1.34
Equity	0.26	3.86	-3.57	4.57	-0.53	-0.84	-1.25
LV	0.15	-2.81	5.34	-4.77	0.76		
VRP	-0.44	2.13	-3.82	10.37	-0.82	-1.05	

	Equity* LV^2	Equity* VRP^2	Bond^2* Equity	Bond^2* LV	Equity^2* VRP	LV^2* VRP	Equity*LV* VRP
Bonds							0.86
Equity	3.04	3.00					
LV		-2.87	-0.97		-2.31		
VRP	2.76		0.69	-0.92		3.68	

Appendix 4: Optimization results

Table 5
Portfolio allocation: Minimum Modified VaR
US, February 1990 – August 2008

	Bond Equity	Bond Equity + LV	Bond Equity + VRP	Bond Equity + LV + VRP
Mean Ann. Return	8.19%	8.65%	14.89%	15.68%
Ann. Std. Dev.	5.41%	4.66%	5.29%	4.11%
Skewness	-0.04	0.47	-0.41	-0.04
Kurtosis	4.15	3.93	3.44	4.24
Min Monthly Loss	-4.02%	-3.12%	-3.52%	-3.19%
Max Monthly Gain	5.98%	5.57%	4.75%	4.77%
Mod. VaR(95%)	1.87%	1.28%	1.43%	0.63%
Sharpe Ratio	0.68	0.89	1.96	2.72
Success Rate	71.3%	70.0%	80.3%	89.2%
<i>Bond</i>	79%	50%	78%	40%
<i>Equity</i>	21%	31%	2%	16%
<i>LV</i>	-	19%	-	24%
<i>VRP</i>	-	-	20%	20%

Table 6
Portfolio allocation: Modified VaR 2%
US, February 1990 – August 2008

	Bond Equity	Bond Equity + LV	Bond Equity + VRP	Bond Equity + LV + VRP
Mean Ann. Return	8.48%	9.54%	22.26%	30.89%
Ann. Std. Dev.	5.85%	6.37%	7.14%	8.51%
Skewness	0.08	0.41	-1.10	-1.40
Kurtosis	3.93	3.26	6.20	10.07
Min Monthly Loss	-4.25%	-3.25%	-8.59%	-12.89%
Max Monthly Gain	6.70%	6.54%	6.52%	8.66%
Mod. VaR(95%)	2.00%	2.00%	2.00%	2.00%
Sharpe Ratio	0.68	0.79	2.49	3.10
Success Rate	68.2%	65.5%	87.0%	89.2%
<i>Bond</i>	69%	6%	60%	0%
<i>Equity</i>	31%	56%	0%	0%
<i>LV</i>	-	38%	-	38%
<i>VRP</i>	-	-	40%	62%

Table 7
Portfolio allocation: Modified VaR 4%
US, February 1990 – August 2008

	Bond Equity	Bond Equity + LV	Bond Equity + VRP	Bond Equity + LV + VRP
Mean Ann. Return	9.57%	10.12%	34.22%	39.36%
Ann. Std. Dev.	9.74%	9.95%	12.23%	12.94%
Skewness	-0.26	-0.18	-1.71	-1.94
Kurtosis	3.58	3.46	9.71	11.70
Min Monthly Loss	-9.07%	-7.89%	-18.05%	-20.65%
Max Monthly Gain	9.37%	9.39%	10.76%	11.32%
Mod. VaR(95%)	4.00%	4.00%	4.00%	4.00%
Sharpe Ratio	0.52	0.56	2.43	2.69
Success Rate	65.9%	64.6%	85.7%	86.5%
<i>Bond</i>	<i>30%</i>	<i>0%</i>	<i>28%</i>	<i>0%</i>
<i>Equity</i>	<i>70%</i>	<i>84%</i>	<i>0%</i>	<i>0%</i>
<i>LV</i>	<i>-</i>	<i>16%</i>	<i>-</i>	<i>16%</i>
<i>VRP</i>	<i>-</i>	<i>-</i>	<i>72%</i>	<i>84%</i>